Unit III

5. (a) Let K be an extension field of F and $a \in$ K. Define adjoin field F(a) of F by the element a. Prove that either :

$$F(a) \cong F(x) = \{f(x)/g(x) : f(x), g(x) \in F[x], g(x) \neq 0\} \quad or \quad F(a) \cong \frac{F[x]}{\langle p(x) \rangle}$$

for some irreducible polynomial $p(x) \in F[x]$.

(b) Let K be an extension field of F and $E = \{a \in K : a$ is algebraic over F$\}$. Then prove that E is an algebraic extension over F. If F is a finite field, then comment on the existence of the field E.

6. (a) Let P be a prime field. Then prove either $P \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p *or* $P \cong \mathbb{Q}$.

(b) If F is a field of characteristic 0, and a and b are algebraic over F, then prove that there is an element $c \in F(a, b)$ such that $F(a, b) = F(c)$.

AA-311

M. Sc. EXAMINATION, Dec. 2017

(First Semester)

(Main & Re-appear)

MATHEMATICS

MAT-501-B

Algebra

Time : 3 Hours] [*Maximum Marks :* 100

Before answering the question-paper candidates should ensure that they have been supplied to correct and complete question-paper. No complaint, in this regard, will be entertained after the examination.

Note : Attempt *Five* questions in all, selecting at least *one* question from each Unit. All questions carry equal marks.

Unit I

1. (a) Define composition series. Prove that every finite group has a composition series.

   (b) Define refinements of a series of a group. Prove that any two subnormal series of a group have equivalent refinements.

2. (a) Let G be a multiplicative group. If H and K be two subgroup of G. Then of following :

   (i) $[H, K] = [K, H]$

   (ii) If H is normal in G, then $[H, K]$ is a subgroup of H.

   (iii) If H and K are normal is G, the $[H, K]$ is a normal subgroup in G such that $[H, K] \subseteq H \cap K$. Deduce that if H is normal in G, then $[H, G]$ is normal G such that $[H, G] \subseteq H$.

   (b) State and prove Jordon-Holder Theorem for finite groups.

Unit II

3. (a) Define a nilpotent group and its class of nilpotency. If G is nilpotent group, then prove that $Z(G) \neq \{e\}$. Further, show that S_3 is not nilpotent.

   (b) Let H be a proper subgroup of a nilpotent group G. Then, prove that is a proper subgroup of its normalizer. Further, if H is a maximal subgroup of G, then prove that it is a normal subgroup of G.

4. (a) Let G be a solvable group and H be a normal subgroup of G. Then prove that H and G|H both are solvable.

   (b) Prove that a group G is solvable if and only if $G^{(k)} = \{e\}$ for some positive integer k, where $G^{(k)} = (G^{(k-1)})$ is the derived subgroup of $G^{(k-1)}$. Deduce that the solvability of S_n for any $n \geq 1$.

Unit IV

7. (a) State the fundamental theorem of Galois theory. Prove that the Galois group of a Galois extension $\mathbb{Q}\left(\sqrt{3},\sqrt{5}\right)$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) Let K be a normal extension of F and let H be a subgroup of G(K, F), where G(K, F) is the set of all automorphisms of K leaving every element of F fixed, let :

$$K_H = \{x \in K : \sigma(x) = x \text{ for all } \sigma \in H\}$$

be the fixed field of H. Then prove that :

(i) $[K : K_H] = O(H)$

(ii) $H = G(K, K_H)$.

8. (a) Show that it is impossible to trisect 60º by ruler and compass.

(b) If $p(x) \in F[x]$ is solvable by radicals over F, then show that the Galois group over F of $p(x)$ is a solvable group.

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