(b) Define norm of an element of $Q(\sqrt{m})$.

If α is an algebraic integer of $Q(\sqrt{m})$, then prove that $N(\alpha)$ is a rational integer. **5**

(c) Prove that an algebraic integer $\in$ of a quadratic field is a unity iff $N(\in) \pm 1$. **5**

4. (a) Define Euclidean field. Prove that the field $Q(i)$ is Euclidean. **10**

(b) Obtain all the unities of the field $Q(i)$. **10**

Unit III

5. (a) Define prime in a quadratic field. Prove that primes of $Q(\sqrt{2})$ are given by :

(i) $\sqrt{2}$ and its associates

(ii) The rational primes of the form $8n \pm 3$ and their associates.

(iii) The factors $a + b\sqrt{2}$ of rational primes of the form $8n \pm 1$

DD-314

M. Sc. EXAMINATION, May 2017

(Fourth Semester)

(Main & Re-appear)

MAT-610-B

MATHEMATICS

ANALYTICAL NUMBER THEORY-II

Time : 3 *Hours*]   [*Maximum Marks :* 100

Before answering the question-paper candidates should ensure that they have been supplied to correct and complete question-paper. No complaint, in this regard, will be entertained after the examination.

Note : Attempt *Five* questions in all, selecting at least *one* question from each Unit. All questions carry equal marks.

Unit I

1. (a) Define Riemann Zeta function and prove that if $s > 1$, then

$$\zeta(s) = \prod_{p}\left(\frac{1}{1-p^{-s}}\right)$$

where the product is over all primes p.

(b) For each integer $s \geq 2$, let P(s) denote the probability that s randomly and independently chosen integers have greatest common divisor 1. Then show that :

$$p(s) = \prod_{p}\left(1-p^{-s}\right).$$ Also show that

$$p(z) = \frac{6}{\pi^2}.$$ **10,10**

2. (a) Let λ be a function defined by $\lambda\left(p_1^{e_1} p_2^{e_2}........p_k^{e_k}\right) = (-1)^{e_1+e_2+....e_k}$ and $\lambda(1) = 1$, where $p_1, p_2,......p_k$ are distinct primes. Then show that :

$$\sum_{d|n}\lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a perfect square} \\ 0 & \text{otherwise} \end{cases}$$

Hence show that $\displaystyle\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}\,\forall s > 1$.

(b) Define Euler product. If f is a multiplicative arithmetic function, and $\displaystyle\sum_{n=1}^{\infty}f(n)$ is absolutely convergent, then prove that :

$$\sum_{n=1}^{\infty}f(n) = \prod_{p}\left(1+f(p)+f\left(p^2\right)+........\right),$$

where p ranges over all the primes. **10,10**

Unit II

3. (a) Prove that the algebraic integers of $Q\sqrt{m}$ are of the type $a+b\sqrt{m}$ and $a + b\tau$, where $\tau = \dfrac{\sqrt{m}-1}{2}$, $a, b \in Z$ if $m \equiv 2$ or $3(\text{mod } 4)$ and $m \equiv 1(\text{mod } 4)$ respectively.

10

Unit IV

7. (a) State Mobius inversion formula and using this prove that :

$$\sum_{d\mid n}\tau(d)\mu\left(\frac{n}{d}\right)=\sum_{d\mid n}\mu(d)\tau\left(\frac{n}{d}\right)=1$$

where $n \geq 1$ also τ denotes the divisor function. Verify this equation for $n = 12$ also. **5**

(b) Show that Mobius function μ is multiplicative. **5**

(c) Prove that the average order of $\tau(n)$ is $n \log_e n$. **10**

8. (a) Define $\sigma(n)$ and show that : **5**

$$\sigma(4m-1) \equiv 0(\mod 4)$$

and $\sigma(12m\ 1) \equiv 0(\mod 12)$.

(b) Let G denote the set of all arithmetic functions for which $f(1) \neq 0$. Then prove that G is an Abelian group w.r.t. dirichlet product. **15**

(b) Let π be a prime of Q(i) with odd norm and let $(d, \pi) = 1$ in Q(i). Then prove that :

$$\alpha^{\phi(\pi)} \equiv 1(\mod \pi) \qquad \textbf{10,10}$$

6. (a) Define Fibonacci number u_n and Lucas number v_n. Prove that :

(i) $u_m v_n + u_n v_m = 2u_{m+n}$

(ii) $v_n^2 - 5u_n^2 = (-1)^n\,4$

(iii) $u_n^2 - u_{n-1}u_{n+1} = (-1)^{n-1}$

(iv) $v_n^2 - v_{n-1}v_{n+1} = 5(-1)^n$

Hence show that :

$$(u_n, u_{n+1}) = (v_n, u_{n+1}) = 1$$

and $u_n\mid u_{rn} \ \forall\ r \geq 1$.

(b) Prove that the Euclidean Algorithm is equivalent to the following condition in $Q(\sqrt{m})$. Given any δ (algebraic integer or not) of $Q(\sqrt{m})$, $\exists$ an algebraic integer K s.t.

$$|N(s - K)| < 1 \qquad \textbf{10,10}$$