Unit III

5. (a) Let $a \in$ K be an algebraic element over F. Then prove that there exists a unique monic irreducible polynomial $p(x) \in$ F[x] such that $p(a) = 0$. Further prove that F(a) is an intension field of F that contains a and [F(a) : F] = degree of $p(x)$.

(b) If K is a finite extension over L and L is a finite extension over F, then prove that K is also a finite extension over F.

6. (a) Define splitting field of a polynomial over a field. Find the splitting field of the polynomial $x^4 + 1$ and the degree of extension over $\mathbb{Q}$, the field of rational numbers.

(b) If $p(x)$ is a polynomial in F[x] of degree $n \geq 1$ and is irreducible over F, then prove that there is an extension E of F, such that [E, F] = n, in which $p(x)$ has a root.

GG-341

B. Sc. (Hons)/M. Sc.

EXAMINATION, Dec. 2017

(Seventh Semester)

(Dual Degree) (Main & Re-appear)

MATHEMATICS

MAT-511-H

Advanced Abstract Algebra

Time : 3 *Hours*] [*Maximum Marks :* 75

Before answering the question-paper candidates should ensure that they have been supplied to correct and complete question-paper. No complaint, in this regard, will be entertained after the examination.

Note : Attempt *Five* questions in all, selecting at least *one* question from each Unit. All questions carry equal marks.

Unit I

1. (a) Let G be a finite group. Then prove that :

$$O(G) = O(Z(G)) + \sum_{a \notin Z(G)} \frac{O(G)}{O(N(a))},$$

where O(A) denotes the number of elements in A.

(b) Let G be a finite group of the order $p^m q$, where p, q are primes and integer $m \geq 1$. Then prove that there exists a subgroup H of G of the order p^m.

2. (a) If $a, b, c \in G$, then show that :

(i) $[a, b, c] = e$ if and only if $[a, b]^c = [a, b]$.

(ii) $[ab, c] = [a, c]^b [b, c]$ and $[a, bc] = [a, c][a, b]^c$

(iii) $[a, b^{-1}, c]^b [b, c^{-1}, a]^c [c, a^{-1}, b]^a = e$.

(b) State and prove Jordan-Holder theorem for arbitrary groups.

Unit II

3. (a) Define nilpotent and solvable groups. Prove that every nilpotent group is solvable. Give an example of a solvable group which is not nilpotent.

(b) Define lower and upper central series of a group. Prove that if group G is nilpotent of class c, then :

(i) $Z_c(G) = G$ and $Z_{c-1}(G) \neq G$,

(ii) $\gamma_{c+1}(G) = \{c\}$ and $\gamma_c(G) \neq \{e\}$.

4. (a) Let G be a solvable group and H be a normal subgroup of G. Then prove that H and G|H both are solvable.

(b) Prove that S_n is not solvable for any integer $n \geq 5$.

Unit IV

7. (a) Define normal extension. Prove that every finite normal extension is the splitting field of some polynomial.

(b) Show that the set of non-zero element of a finite field forms a multiplicative cyclic group.

8. (a) Prove that the Galois group of $x^3 - 2$ over Q is isomorphic to S_3, the symmetric group of degree 3.

(b) Show that it is impossible to duplicate the cube by ruler and compass.

Unit IV

7. (a) Define normal extension. Prove that every finite normal extension is the splitting field of some polynomial.

(b) Show that the set of non-zero element of a finite field forms a multiplicative cyclic group.

8. (a) Prove that the Galois group of $x^3 - 2$ over Q is isomorphic to S_3, the symmetric group of degree 3.

(b) Show that it is impossible to duplicate the cube by ruler and compass.