# FF342

## B. Sc. (Hons.)-M. Sc. (Mathematics) EXAMINATION, 2020

(Sixth Semester)

(5 Year Integrated Course)

(B Scheme) (Re-appear)

MAT414H

LINEAR ALGEBRA

*Time : 2½ Hours*] [*Maximum Marks :* 75

Before answering the question-paper candidates should ensure that they have been supplied to correct and complete question-paper. No complaint, in this regard, will be entertained after the examination.

**Note** : Attempt *Four* questions in all. All questions carry equal marks.

**1.** (a) Let V = $\{a_0 + a_1x + ........ + a_nx^n : a_i \in F \text{ for } 0 \leq i \leq n, n \text{ is an integer}\}$, the set of all polynomials having coefficients in field F. Show that (V, +, $\cdot$) is a vector space over F where the binary operations vector addition (+) and scalar multiplication ($\cdot$) are referred as the addition of polynomials and the product of a polynomial by an element of F respectively.

(b) Let F be a field, V = $\{(x_1, x_2, .........,x_n) : x_i \in F \text{ for } 1 \leq i \leq n\}$ be a vector space over F and W be the set of all solutions $(x_1, x_2, .........,x_n)$ of a pair of linear equations :

$$a_1x_1 + a_2x_2 + ........ + a_nx_n = 0 \text{ and } b_1x_1 + b_2x_2 + ........ + b_nx_n = 0.$$

Show that W is a subspace of V over the field F, where $a_i$, $b_i \in$ F for $1 \leq i \leq n$.

2. (a) Let W be a subspace of an $n$-dimensional vector space V over a field F such that $(v_1, v_2, .........,v_k)$ is a linearly independent subset of V over F such that $v_i \angle$ W for all $1 \leq i \leq k$. Then prove that the set $\{W + v_i : 1 \leq i \leq k\}$ forms a basis of the quotient space V/W. Deduce that the dimension of W is $n - k$.

(b) Define the sum of two subspaces of a vector space. If $W_1$ and $W_2$ are two subspaces of a vector space V over a field F. Show that $W_1 + W_2$ is also a subspace of V generated by $W_1 \cup W_2$.

3. (a) Find a linear transformation which maps (1, 1, 1), (1, 1, 0), (1, 0, 0) in $R^3$ to (2, 1), (2, 1), (2, 1) in $R^2$. Also find the null space of the transformation. Is the transformation one-one ?

(b) For a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(x, y) = (x - y, -x + y, -x)$, Find a basis and dimension of its range space and its null space.

4. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + y, x - y, 2x + z)$. Find the range space, null space, rank and nullity of T and verify that Rank(T) + Nullity(T) = 3.

(b) Define dual space of a vector space. If S = {(–1, 1, 1), (1, –1, 1), (1, 1, –1)}, then find the dual basis of S.

5. (a) Prove that the mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x, y) = (x - y, -x + 2y, -x)$ is a linear transformation. Also write the associated matrix. Is T singular ? Justify your reply.

(b) A linear transformation $T : U \to V$ of vector space U and V over the same field F is non-singular if and only if T maps every linearly independent subset of U onto a linearly independent subset of V.

6. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defiend by $T(x, y, z) = (2y + z, x - 4y, 3x)$. Find the matrix of T with respect to basis B = {(1, 1, 1), (1, 1, 0), (1, 0, 0)). Also verify $[T, B][u, B] = [T(u), B]$ for all $u \in \mathbb{R}^3$.

(b) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y, z) = (x + y, + z, 2y + z, 2y + 3z)$. Find the eigen values and the basis for the eigen space.

7. (a) State and prove Bessel's inequality.

(b) Prove that every finite dimensional vector space is an inner product space.

8. (a) Define a self-adjoin operator. If A and B are linear operators on a finite dimensional inner product space V(F). Prove that AB = BA if and only if $A' = A$ and $B' = B$.

(b) Let S be the subspace of an inner product space $\mathbb{R}^4$ spanned by the vectors $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 2, 4, 5)$, $v_3 = (1, -3, -4, -2)$ in $\mathbb{R}^4$. Apply the Gram-Schmidt orthogonal process to find an orthogonal basis and then an orthonormal basis of S.