Roll No. ....

18AA1901

M. Sc. EXAMINATION, 2021

(First Semester)

(C. Scheme) (Main & Re-appear)

MATHEMATICS

MAT501C

Abstract Algebra-I

Time : 2½ Hours] [*Maximum Marks :* 75

Before answering the question-paper candidates should ensure that they have been supplied to correct and complete question-paper. No complaint, in this regard, will be entertained after the examination.

Note : Attempt any *Four* questions. All questions carry equal marks.

1. (a) Derive the class equation of a finite group G.

(b) State and prove P. Hall Lemma for three subgroups.

2. (a) If G is a finite group of order n and p is a prime such that p divides n, then prove that G has a non-identity element of order p. If the order of G is pq, where p and q are primes, then show that G is not a simple group.

(b) State and prove Sylow's third theorem for a finite group G.

3. (a) Prove that an abelian group has a composition series if and only if it is finite. Explain, why the group $(\mathbb{Z}; +)$ of all integers has no composition series ?

(b) Prove that any *two* composition series of a group G are equivalent. Hence show that every group having a composition series determines a unique list of simple groups.

4. (a) In a nilpotent group G, prove that the following :

   (i) Z(G) is non-trivial subgroup of G.

   (ii) $H \cap Z(G) \neq \{e\}$ for every normal subgroup H of G.

   (b) Prove that G is solvable if and only if $G^{(n)} = \{e\}$ for some positive integer n.

5. (a) Define prime subfield P. Show that $P \cong \mathbb{Q}$ or $P \cong \mathbb{Z}_p$ for some prime p. Deduce that the characteristic of a finite field is a prime number.

   (b) Suppose that E is an extension field of F and $\alpha, \beta \in E$ are algebraic elements of degree m and n over F, respectively, with $\gcd(m, n) = 1$. Prove that $F(\alpha, \beta)$ is a finite extension over F with $[F(\alpha, \beta) : F] = mn$.

6. (a) Let $f(x)$ be a polynomial of degree $n \geq 1$ over a field F. Then prove that there exists an extension field K of F which contains a root of $f(x)$.

   (b) Define the splitting field of a polynomial over a field. Determine the degrees of the splitting fields of the following polynomials $x^6 + x^3 + 1$ and $x^4 + 1$ over Q.

7. (a) Define separable extension. If K is a finite and separable extension of F. Show that K is a simple extension of F.

   (b) Define normal extension. If K is an extension field of F such that $[K : F] = 2$, then show that K is a normal extension of F. Further, prove that every normal extension of F is the splitting field of some polynomial over F.

8. (a) Let K be a finite Galois extension field of a field F and let $G = G(K/F)$, the Galois group. Define the fixed field K_H of a subgroup H of G. Show that K_H is a subfield of K. Further, if $f(x)$ is an irreducible polynomial over F and $\sigma \in G$, then prove that $a \in K$ is a root of $f(x)$ if and only if $\sigma(a)$ is a root of $f(x)$.

   (b) State and prove the fundamental theorem of Galois theory.