(b) Prove that the primes of $Q\left(\sqrt{-3}\right)$ are given by : **10**

(i) $1 - w$ and its associates

(ii) The rational primes $3x + 2$ and its associates.

(iii) The factors $a + bw$ of rational primes of the form $3n + 1$.

where w is primitive cube root of unity.

Unit III

5. (a) Find the unities of the field $Q\left(\sqrt{2}\right)$. **10**

(b) Prove that every non-zero, non-unity algebraic integer of $Q\left(\sqrt{m}\right)$ can be expressed as a product of primes in $Q\left(\sqrt{m}\right)$. Give an example of a quadratic field in which uniqueness of this factorization does not hold. **10**

DD-314

M. Sc. EXAMINATION, May 2018

(Fourth Semester)

(Main & Re-appear)

MATHEMATICS

MAT610B

ANALYTICAL NUMBER THEORY-II

Time : 3 *Hours*] [*Maximum Marks* : 100

Before answering the question-paper candidates should ensure that they have been supplied to correct and complete question-paper. No complaint, in this regard, will be entertained after the examination.

Note : Attempt *Five* questions in all, selecting at least *one* question from each Unit. All questions carry equal marks.

Unit I

1. (a) Define Riemann Zeta function and prove that if $s > 1$, then $\zeta(s) = \prod_{p}\left(\frac{1}{1+p^{-s}}\right)$, where the product is over all primes p. **10**

   (b) For each integer $s \geq 2$, let P(s) denote the probability that s randomly and independently chosen integers have gcd equal to 1. Then prove that : **10**

$$P(s) = \frac{1}{\zeta(s)}$$

2. (a) Define Dirachlet series. Suppose $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$, $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ and $H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$, where $h = f * g$. Then prove that $H(s) = F(s)G(s)$ for all s such that F(s) and G(s) both converge absolutely. Hense show that : **10**

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

   (b) Prove that : **10**

$$\zeta(2K) = \frac{(-1)^{K-1} 2^{2K-1} \pi^{2K} B_{2K}}{(2K)!}$$

Unit II

3. (a) Define algebraic integer of a quadratic field and prove that algebraic integers of Q(i) one of the form atb_i where $a + b$ are national integers. **10**

   (b) Prove that the set of Gaussian integers is a ring w.r.t. addition and multiplication of complex numbers. **5**

   (c) If α is an algebraic integer of Q(i) and $N(\alpha) = \pm p$, where p is a rotational prime. Then prove that α is a prime of Q(i). **5**

4. (a) Prove that the fields $Q\sqrt{m}$ are Euclidean for $m = -1$ and $m = -2$. **10**

$$g(n) = \sum_{d/n} f(d)\mu\left(\frac{n}{d}\right) = \sum_{d/n} \mu(d) f\left(\frac{n}{d}\right)$$

for all n.

8. (a) Let $\wedge(n)$ be a function defined as :

$$\wedge(n) = \begin{cases} \log_e^p & \text{if } n = p^e \\ 0 & \text{otherwise} \end{cases}$$

where p is a prime and integer $e > 0$. Then show that $\sum_{d/n} \wedge(d) = \log_e^n$ and hence

$$\wedge(n) = \sum_{d/n} \log_e^d \mu\left(\frac{n}{d}\right) = -\sum_{d/n} \log_e^d \mu(d).$$

10

(b) Prove that for all $n < 1$, $\exists$ constant A such that $A < \frac{\sigma(n)\phi(n)}{n^2} < 1$. **10**

6. (a) Let π be a prime of Q(i) with add norm and let α be an algebraic integer of Q(i) s.t. $(\alpha, \pi) = 1$. Then prove that : **15**

$$\alpha^{\phi(\pi)} \equiv (\text{mod}\, \pi).$$

(b) Define Fibonacci numbers u_n and Lucas numbers vn and prove that : **5**

$$u_{p-1} \equiv 0(\text{mod}\, p) \text{ if } p = 5n \pm 1.$$

Unit IV

7. (a) Define multiplicative function. If g is a multiplicative function and $f(n) = \sum_{d/n} g(d)$ for all n, then prove that f is also multiplicative. Hence show that $\sigma(n)$ is multiplicative. **10**

(b) Let f and g be be arithmetic functions. Then prove that $f(n) = \sum_{d/n} g(d)$ for all n iff : **10**